PMM U.S.S.R., Vol.54, No.6, pp.756-760, 1990
Printed in Great Britain

0021-8928/90 $10.00+0.00

THE PRINCIPLE OF LEAST CONSTRAINT FOR SYSTEMS WITH NON-RESTORING CONSTRAINTS*

V.A. SINITSYN

Two new versions of the principle of least constraint are derived from the D'Alembert-Lagrange principle for systems with ideal holonomic and non-holonomic restoring and non-restoring constraints. The first version is similar to Boltzmann's and Bolotov's modification of Gauss's principle for systems with non-restoring constraints. The difference is that here the actual motion is determined in a certain bounded set of possible motions as the one that deviates least from the motion of the system with all non-restoring constraints and any part of the restoring constraints disengaged. According to the second version of the principle, the actual motion is found by comparing certain distinguished possible motions as to their deviation from the motion of the system obtained by eliminating any part of the non-restoring and any part of the restoring constraints. Examples are given.

1. Consider the motion of a mechanical system with ideal holonomic and non-holonomic constraints, some of which are restoring and some non-restoring:

$$f_s(\mathbf{r}_1, \mathbf{v}_1, \ldots, \mathbf{r}_n, \mathbf{v}_n, t) = 0 \quad (s = 1, \ldots, l)$$
$$f_s(\mathbf{r}_1, \mathbf{v}_1, \ldots, \mathbf{r}_n, \mathbf{v}_n, t) \geqslant 0 \quad (s = l+1, \ldots, r) \tag{1.1}$$

$\mathbf{r}_1, \ldots, \mathbf{r}_n, \mathbf{v}_1, \ldots, \mathbf{v}_n$ are the radius-vectors and velocity vectors of the material points of the system, with numbers $k = 1, \ldots, n$; f_s are continuous and differentiable functions (twice differentiable for the holonomic constraints) of their arguments: the time t, coordinates x_k, y_k, z_k and velocities v_{kx}, v_{ky}, v_{kz}.

The principle of least constraint (Gauss's principle) will be derived, following Bolotov /1/, from the D'Alembert-Lagrange principle for systems with non-restoring constraints

$$\Sigma (\mathbf{F}_k - m_k \mathbf{w}_k)\, \delta\mathbf{r}_k \leqslant 0 \tag{1.2}$$

where $\mathbf{F}_k$ are the active forces, m_k and $\mathbf{w}_k$ are the masses and accelerations of the material points, and $\delta\mathbf{r}_k$ are the virtual displacement at time t in the state defined by radius-vectors $\mathbf{r}_k$ and velocity vectors $\mathbf{v}_k$. Here and below summation over k is from $k = 1$ to $k = n$.

Under the constraints (1.1), the virtual displacements $\delta\mathbf{r}_k$ satisfy the conditions

$$\Sigma \mathbf{N}_{sk} \delta\mathbf{r}_k = 0 \ (s = 1, \ldots, l), \ \Sigma \mathbf{N}_{sk} \delta\mathbf{r}_k \geqslant 0 \quad (s = l+1, \ldots, r) \tag{1.3}$$

where $\mathbf{N}_{sk}$ $(s = 1, \ldots, r;\ k = 1, \ldots, n)$ are continuous vector-valued functions of the coordinates and velocities of the points and the time: $\mathbf{N}_{sk} = (\partial f_s/\partial x_k,\ \partial f_s/\partial y_k,\ \partial f_s/\partial z_k)^T$ for the holonomic constraints and $\mathbf{N}_{sk} = (\partial f_s/\partial v_{kx},\ \partial f_s/\partial v_{ky},\ \partial f_s/\partial v_{kz})^T$ for the non-holonomic constraints.

Non-restoring constraints which may be reduced as to position and/or velocity are not taken into consideration, since when the forces are bounded in magnitude there is a time interval during which they do not obstruct the motion. Consideration is given only to non-restoring constraints which may be strained /2/ in a given state $(f_s(\mathbf{r}_1, \mathbf{v}_1, \ldots, \mathbf{r}_n, \mathbf{v}_n, t) = 0,\ s = 1, \ldots, r)$, i.e., they may create reactions and change the accelerations of points, co-ordinating them with the imposed constraints. Thus, the restrictions due to restoring and non-restoring constraints (1.1) are satisfied provided the following equations and inequalities, which are linear in the accelerations, hold:

$$\alpha_s = 0 \ (s = 1, \ldots, l), \ \alpha_s \geqslant 0 \quad (s = l+1, \ldots, r < 3n)$$
$$\alpha_s = \Sigma \mathbf{N}_{sk} \mathbf{w}_k + d_s \ (s = 1, \ldots, r) \tag{1.4}$$

d_s $(s = 1, \ldots, r)$ are continuous functions of the coordinates and velocities of the points and

*Prikl. Matem. Mekhan., 54, 6, 920-925, 1990 of the time, l is the number of restoring constraints, and $(r - l)$ is the number of non-restoring constraints; for brevity, the non-negative quantities α_s $(s = 1, \ldots, r)$ will be referred to as the constraint-reducing accelerations.

To virtual displacements which satisfy conditions (1.3) there correspond possible motions with possible accelerations $\mathbf{w}_k'$ (i.e., accelerations consistent with the constraints; $\mathbf{w}_k$ are the accelerations of the material points in the actual motion) /3/:

$$\delta \mathbf{r}_k = {}^1/_2\, (\mathbf{w}_k' - \mathbf{w}_k)(\mathrm{d}t)^2 \tag{1.5}$$

This correspondence will be established if we can define motions with possible accelerations such that the constraint-reducing accelerations (α_s') are not less than the constraint-reducing accelerations in the actual motion (α_s), i.e.,

$$\alpha_s' = \alpha_s = 0 \quad (s = 1, \ldots, l), \ \ \alpha_s' \geqslant \alpha_s \geqslant 0 \quad (s = l + 1, \ldots, r) \tag{1.6}$$

$$\alpha_s' = \Sigma \mathbf{N}_{sk} \mathbf{w}_k' + d_s \quad (s = 1, \ldots, r)$$

Then the differences $(\mathbf{w}_k' - \mathbf{w}_k)$ in (1.5) for the possible accelerations determined by (1.6) will satisfy the conditions

$$\Sigma \mathbf{N}_{sk} (\mathbf{w}_k' - \mathbf{w}_k) = 0 \quad (s = 1, \ldots, l), \ \ \Sigma \mathbf{N}_{sk} (\mathbf{w}_k' - \mathbf{w}_k) \geqslant 0 \quad (s = l + 1, \ldots, r). \tag{1.7}$$

which have the same form as conditions (1.3) imposed on the virtual displacements $\delta \mathbf{r}_k$.

Substituting (1.5) into (1.2), we obtain the following inequality for the distinguished possible motions:

$$\Sigma\, (\mathbf{F}_k - m_k \mathbf{w}_k)(\mathbf{w}_k' - \mathbf{w}_k) \leqslant 0 \tag{1.8}$$

Now consider a "released" system, i.e., a system obtained from the original one by eliminating all the non-restoring constraints and any part of the restoring ones. Letting $\mathbf{w}_k^0$ denote the accelerations of the material points of the released system in its actual motion, given that the active forces and the initial state are the same, we can write the general equation of analytical dynamics as follows:

$$\Sigma\, (\mathbf{F}_k - m_k \mathbf{w}_k^\circ)\, \delta \mathbf{r}_k = 0 \tag{1.9}$$

The vertical displacements $\delta \mathbf{r}_k$ in inequality (1.2) (which satisfy (1.3)) are also virtual displacements of the released system and may be inserted into Eq.(1.9). Replacing them in (1.9) by their expressions from (1.5), we obtain the equation

$$\Sigma\, (\mathbf{F}_k - m_k \mathbf{w}_k^\circ)(\mathbf{w}_k' - \mathbf{w}_k) = 0$$

Subtracting this equation from (1.8), we obtain the inequality

$$\Sigma m_k\, (\mathbf{w}_k^\circ - \mathbf{w}_k)(\mathbf{w}_k' - \mathbf{w}_k) \leqslant 0$$

Using a standard property of bilinear forms, we can rewrite this inequality in the form

$$S_{\delta d} - S_{\delta o} + S_{do} \leqslant 0 \tag{1.10}$$

where $S_{\delta d}$ is the deviation of the distinguished possible (δ) motion from the actual (d) motion of the system, $S_{\delta o}$ is the deviation of the distinguished possible motion from the actual motion of the released system (o), and S_{do} is the deviation of the actual motion from the actual motion of the released system:

$$S_{\delta d} = \frac{1}{2}\, \Sigma m_k\, (\mathbf{w}_k' - \mathbf{w}_k)^2, \qquad S_{\delta o} = \frac{1}{2}\, \Sigma m_k\, (\mathbf{w}_k' - \mathbf{w}_k^\circ)^2$$

$$S_{do} = \frac{1}{2}\, \Sigma m_k\, (\mathbf{w}_k - \mathbf{w}_k^\circ)^2$$

Since all the terms on the left of (1.10) are non-negative and vanish only when the accelerations of the points in the respective motions are identical, we also have

$$S_{do} < S_{\delta o} \tag{1.11}$$

This inequality states the principle of least constraint for systems with non-restoring constraints, in the following form: the deviation of the actual motion from the actual motion of the released system obtained by removing all non-restoring and some of the restoring constraints is less than the analogous deviation of any possible motion for which the constraint-slackening accelerations do not fall below their values in the actual motion.

A brief discussion now follows of the difference between the assertion just proved and the formulation of the modified Gauss principle as substantiated by Boltzmann and Bolotov

/1/: the deviation of the actual motion of the system from its actual motion when all non-restoring constraints and an arbitrary number of restoring constraints are eliminated is less than the deviation of any of the possible motions. Even before the appearance of /1/, authors engaged in investigating systems with non-restoring constraints were aware of the difficulty of deriving Gauss's principle from the D'Alembert-Lagrange principle and the obscurity of Boltzmann's exposition of the question in /4/. Bolotov /1/ pointed out that the derivation relied on the following underlying assumption: inequality (1.2) remains valid if δr_k is replaced by any virtual displacements which satisfy only those of inequalities (1.3) associated with constraints which are not slackened at time t in the actual motion.

Hence it is evident that the modification of Gauss's principle as formulated involves possible motions of a new system, obtained by eliminating those non-restoring constraints which are slackened in the actual motion (the slackening accelerations in the actual motion are greater than zero). Generally speaking, it is not known in advance which of the non-restoring constraints are slackened as to acceleration, and their determination is a major problem in connection with the motion of systems with non-restoring constraints. Nevertheless, a property of the actual motion follows from this assumption makes it possible to pick out the actual motion among the possible motions selected for comparison, which generally involve less non-restoring constraints. The version of the principle obtained here picks out the actual motion from a more restricted set of possible motions.

Example 1. Consider a mathematical pendulum on a flexible cord of length l. In polar coordinates r, φ, with origin at the point of suspension, the condition of releasability from the constraint is $l - r \geqslant 0$. In the "constrained" state we have $r = l$ and $r^{\cdot} = 0$. Using the principle of least constraint, let us find the condition for the cord to slacken in respect of its acceleration ($r^{\cdot\cdot} < 0$).

We first construct the function S_{δ_0} (see Sect.1) (m is the mass of the material point) ($w_r = r^{\cdot\cdot} - r\varphi^{\cdot 2}$, $w_\varphi = r\varphi^{\cdot\cdot} + 2r^{\cdot}\varphi^{\cdot}$):

$$2S_{\delta_0} = m\,[(w_r' - w_r^\circ)^2 + l^2 (w_\varphi' - w_\varphi^\circ)^2]$$

A free material point (in the force field of a mathematical pendulum) has the acceleration due to gravity g; therefore, in the state $r, \varphi, r^{\cdot}, \varphi^{\cdot}$ ($r = l$, $r^{\cdot} = 0$), the generalized accelerations of the released material point are:

$$r^{\cdot\cdot 0} = r\varphi^{\cdot 2} + g\cos\varphi,\quad \varphi^{\cdot\cdot 0} = -gl^{-1}\sin\varphi$$

The necessary conditions for the function S_{δ_0} in this example to have a minimum ($\partial S_{\delta_0}/\partial r^{\cdot\cdot\prime} = 0$, $\partial S_{\delta_0}/\partial\varphi^{\cdot\cdot\prime} = 0$) lead to the expressions

$$r^{\cdot\cdot} = l\varphi^{\cdot 2} + g\cos\varphi,\quad \varphi^{\cdot\cdot} = -gl^{-1}\sin\varphi$$

which, in view of the inequality $r^{\cdot\cdot} < 0$ (the slackening condition), imply the following well-known inequality (φ_0, φ_0', $r_0 = l$, $r_0^{\cdot} = 0$ are the initial data):

$$l\varphi_0^{\cdot 2} + g\,(3\cos\varphi - 2\cos\varphi_0) < 0$$

which defines the angle φ at the instant the cord crumples up.

Example 2. Consider a circular, inhomogeneous, absolutely rigid disc of radius R and mass m, rolling without sliding along a straight horizontal track in a uniform gravitational force field (plane motion in a vertical plane). The centre of mass C is at a distance a from the centre O of the disc (we do not require that $a \leqslant R$), the moment of inertia of the disc relative to an axis through the centre of mass and perpendicular to the plane of the motion is equal to J. In the basic coordinate system (one axis along the horizontal guide in the direction of motion of the centre O of the disc and the other vertically upward) the position of the disc is determined by the coordinates x, y of the centre O and the angle of rotation φ (between the vertical axis and the ray OC).

We have one non-restoring and one restoring constraint - the unilateral constraint furnished by the horizontal guide and the condition of rolling without sliding: $y - R \geqslant 0$, $x - R\varphi = 0$. When these conditions are expressed as equalities, and together with them we take $y^{\cdot} = 0$ and $x^{\cdot} - R\varphi^{\cdot} = 0$, the accelerations must satisfy the conditions

$$y^{\cdot\cdot} \geqslant 0,\quad x^{\cdot\cdot} - R\varphi^{\cdot\cdot} = 0 \tag{1.12}$$

We shall find the conditions under which the first constraint of (1.12) in the actual motion is reduced ($y^{\cdot\cdot} > 0$). This will be done on the assumption that the second constraint of (1.12) remains in force. Without discussing the question of whether these constraints - here treated as ideal - are actually realizable, we would like to point out that the situation is quite realistic for some motions when describing the interaction at the point of contact according to the Amontons-Coulomb law of dry friction, which takes the molecular attraction force into consideration (see, e.g., /5/).

The released system is obtained by releasing the disc from the constraints (1.12). For a disc in free plane motion we have

$$x^{\cdot\cdot\circ} = a\varphi^{\cdot 2}\sin\varphi,\quad y^{\cdot\cdot\circ} = -g + a\varphi^{\cdot 2}\cos\varphi,\quad \varphi^{\cdot\cdot\circ} = 0$$

We construct the function S_{δ_0} describing the deviations of the possible motions from the motion of the free disc:

$$2S_{\delta_0} = m\,[(x^{\cdot\cdot\prime} - x^{\cdot\cdot})^{2\circ} + (y^{\cdot\cdot\prime} - y^{\cdot\cdot\circ})^2 + 2a\,(x^{\cdot\cdot\prime} - x^{\cdot\cdot\circ})\,(\varphi^{\cdot\cdot\prime} - \varphi^{\cdot\cdot\circ})\cos\varphi - 2a\,(y^{\cdot\cdot\prime} - y^{\cdot\cdot\circ})\,(\varphi^{\cdot\cdot\prime} - \varphi^{\cdot\cdot\circ})\sin\varphi] + (J + ma^2)\,(\varphi^{\cdot\cdot\prime} - \varphi^{\cdot\cdot\circ})^2$$

The necessary conditions for S_{δ_0} to have a minimum, with the second constraint (1.12) imposed on the possible accelerations, have the following form (λ is an undertermined multiplier):

$$\partial S_{\delta_0}/\partial x^{\cdot\cdot\prime} + \lambda = 0,\ \ \partial S_{\delta_0}/\partial y^{\cdot\cdot\prime} = 0$$
$$\partial S_{\delta_0}/\partial \varphi^{\cdot\cdot\prime} - \lambda R = 0,\ \ x^{\cdot\cdot\prime} - R\varphi^{\cdot\cdot\prime} = 0 \tag{1.13}$$

The actual motion ($y^{\cdot\cdot} > 0$) is determined by the values of $x^{\cdot\cdot}, y^{\cdot\cdot}, \varphi^{\cdot\cdot}, \lambda$ which solve Eqs.(1.13). The partial derivatives in these equations, evaluated for the actual motion (indicated by enclosing the derivatives in question in parentheses), have the following mechanical interpretation: $(\partial S_{\delta_0}/\partial x^{\cdot\cdot\prime})$ is the horizontal component of the reaction at the point of contact; $(\partial S_{\delta_0}/\partial y^{\cdot\cdot\prime}) = m\,(y_c^{\cdot\cdot} + g)$ (where y_c is the coordinate of the centre of mass). To ascertain the mechanical meaning of $(\partial S_{\delta_0}/\partial \varphi^{\cdot\cdot\prime})$, we let O_1 denote the centre of oscillation (a point whose distance l from the centre O equals the reduced length of a physical pendulum such that O is the point of suspension ($|OO_1| = l = a + a_1,\ a_1 = J/(ma)$). Then $(\partial S_{\delta_0}/\partial \varphi^{\cdot\cdot\prime}) = ma\,(w - g\sin\varphi)$, where w is the projection of the absolute acceleration of O_1 on the transverse direction of the polar axes, relative to which O_1 has coordinates l, φ (the centre of the polar system is at O).

The determinant of the set of Eqs.(1.13), as equations in $x^{\cdot\cdot\prime}, y^{\cdot\cdot\prime}, \varphi^{\cdot\cdot\prime}$, for real values of φ, may vanish only if $l = a$ (all the mass concentrated at the point $C, J = 0$), and this happens if $\cos\varphi = -R/a$. In this situation the solution of the problem is not unique (the model is not well-posed).

Eqs.(1.13) imply the following expression for the undetermined multiplier:

$$\lambda = -m\,(l - a)\,\varphi^{\cdot 2}\sin\varphi/\Delta \tag{1.14}$$
$$(\Delta = \sin^2\varphi - 2\nu\cos\varphi - \mu - \nu^2 < 0,\ \ \mu = l/a,\ \ \nu = R/a$$

Using the condition $y^{\cdot\cdot} > 0$, we deduce from the second equation of system (1.13) that

$$a\varphi^{\cdot\cdot}\sin\varphi > g - a\varphi^{\cdot 2}\cos\varphi \tag{1.15}$$

The third equation of system (1.13) may be written as

$$w = g\sin\varphi + \lambda\nu/m \tag{1.16}$$

These conditions express the following properties of the motion of the disc at the instant it becomes detached from the base. If (1.15) is satisfied, we have $(\partial S_{\delta_0}/\partial y^{\cdot\cdot\prime}) \leqslant 0$, when $y^{\cdot\cdot} = 0$ (the constraint is unstrained).

The determinant of system (1.13) increases without limit in magnitude as $a \to 0$ (the centre of mass approaches the centre of the disc). At the same time, inequality (1.15) cannot hold for sufficiently small values of a, and therefore the non-restoring constraint cannot be reduced.

If the first constraint in (1.12) is reduced at $\varphi = 0$, the horizontal component of the reaction is zero (see (1.14)), $a\varphi^{\cdot 2} > g$ (see (1.15)) and $w = 0$. At $\varphi = \pi$ the constraints cannot be reduced, because inequality (1.15) does not hold.

At the instant the first constraint of (1.12) is reduced, w and λ have the same sign as $\sin\varphi$; the projection of the reaction on the horizontal axis has a sign opposite to that of $\sin\varphi$.

If the second constraint of (1.12) is eliminated after the first (non-restoring) constraint has been reduced, then $x^{\cdot\cdot}, y^{\cdot\cdot}, \varphi^{\cdot\cdot}$ take the values $x^{\cdot\cdot\circ}, y^{\cdot\cdot\circ}, \varphi^{\cdot\cdot\circ}$. The disc is detached from the base if $y^{\cdot\cdot\circ} > 0$. But if there is an interval of time during which, besides inequality (1.15), the inequality $y^{\cdot\cdot\circ} < 0$ is also satisfied, one has a process which is characteristic of systems with variable structure.

2. Continuing the development of Mach's concept of least deviation of the actual motion from the motion of a system with a smaller number of constraints (though we should mention that in the context of this statement /6/ there is no mention of non-restoring constraints), we shall now compare the actual motion with possible motions in respect of their deviations from the motion of the system when released from not all but only part of the non-restoring constraints.

Suppose it is proposed to compare motions as to their deviation from the motion of a system in which any part of the restoring and any part of the non-restoring constraints are reduced. Changing the numbering, we number them $s = 1, \ldots, p$.

We wish to determine the possible motions with possible accelerations $\mathbf{w}_k{}'$ $(k = 1, \ldots, n)$ which satisfy the conditions

$$\alpha_s' = \alpha_s \geqslant 0 \quad (s = 1, \ldots, p) \tag{2.1}$$

Let A denote the set of accelerations satisfying conditions (2.1) $(A = \{\mathbf{w}_k' : \alpha_s' = \alpha_s \geqslant 0$ $s = 1, \ldots, p\})$. On substituting the accelerations $\mathbf{w}_k' \in A$ into the constraints numbered $p + 1, \ldots, r$, the latter may be divided into two groups by comparing the appropriate α_s' and α_s $(s = p + 1, \ldots, r)$:

$$\alpha_s' \geqslant \alpha_s \geqslant 0 \quad (s = p + 1, \ldots, q \leqslant r) \tag{2.2}$$

$$\alpha_s > \alpha_s' \geqslant 0 \quad (s = q + 1, \ldots, r) \tag{2.3}$$

Group (2.3) contains the non-restoring constraints which are reduced in the actual motion. Therefore the virtual displacements $\delta\mathbf{r}_k$ in (1.2) do not necessarily satisfy the part of inequalities (1.3) numbered $q + 1, \ldots, r$ (according to the Boltzmann-Bolotov position; see Sect.1). The system with constraints $s = 1, \ldots, p$, the accelerations of whose points ($\mathbf{w}_k^*$ $k = 1, \ldots, n$) satisfy the equalities

$$\alpha_s^* = \alpha_s \quad (s = 1, \ldots, p), \ \alpha_s^* = \Sigma\, \mathbf{N}_{sk}\mathbf{w}_k^* + d_s \tag{2.4}$$

will be called the comparison system.

In view of (1.5) and (2.4), the virtual displacements in the comparison system satisfy the equalities

$$\Sigma \mathbf{N}_{sk}\delta\mathbf{r}_k = 0 \ (s = 1, \ldots, p)$$

The comparison system, therefore, is governed by the following equation of analytical dynamics:

$$\Sigma\, (\mathbf{F}_k - m_k\mathbf{w}_k^*)\, \delta\mathbf{r}_k = 0 \tag{2.5}$$

If the comparison system is used in Sect.1 instead of the released system, one gets the following version of the principle for systems with non-restoring constraints: the deviation of the actual motion of the system from the motion of the comparison system obtained by eliminating $r - p$ restoring and non-restoring constraints with accelerations $\mathbf{w}_k^* \in A = \{\mathbf{w}_k' : \alpha_s' = \alpha_s \geqslant 0 \ \ s = 1, \ \ldots, \ p\}$ is a minimum compared with the deviations of the possible motions with possible accelerations

$$\mathbf{w}_k' \in A \cap B$$

$$B = \{\mathbf{w}_k' : \alpha_s' \geqslant \alpha_s \geqslant 0, \ s = p + 1, \ldots, q \leqslant r\}$$

The Boltlzmann-Bolotov modification of Gauss's principle is a special case of this assertion with $\alpha_s' = \alpha_s = 0$ $(s = 1, \ldots, p)$ (p is the number of restoring constraints after all the non-restoring and part of the restoring constraints have been eliminated), $\alpha_s' \geqslant \alpha_s = 0$ $(s = p + 1, \ldots, q)$.

The version of the principle derived in Sect.1 is also a special case - that with

$$\alpha_s' = \alpha_s = 0 \quad (s = 1, \ldots, p \leqslant l), \ \alpha_s' \geqslant \alpha_s \geqslant 0 \quad (s = p + 1, \ldots r)$$

We note that the derivation of the new versions of the principle remains unchanged if some of the constraints are linearly dependent on the accelerations of the points.

REFERENCES

1. BOLOTOV E.A., On Gauss's principle. Izv. Fiz.-Mat. Obshch. pri Kazan. Univ., Kazan University Press, Kazan, 1916.
2. SUSLOV G.K., Theoretical Mechanics, Gostekhizdat, Moscow-Leningrad, 1944.
3. OSTROGRADSKII M.V., On the principle of virtual velocities and on forces of inertia. Izd. Akad. Nauk SSSR, Leningrad, 1958.
4. BOLTZMANN L., Vorlesungen über die Principe der Mechanik, T. 1, 1897, (/repr.:/ J.A. Barth, Leipzig, 1922).
5. LAVENDEL E.E., (Ed.), Vibrations in Engineering, Mashinostroyeniye, Moscow, 1981.
6. MACH E., Mechanics. Historical-Critical Survey of its Development. "Obschchestvennaya Pol'za" Press, St. Petersburg, 1909.

Translated by D.L.